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Inclusion Results for Cardioid-Type Analytic Functions Based on the Multiplier Transformation

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Abstract

In this paper, we introduce new subclasses $\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\phi_c)$, $\mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_c)$ and $\mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1,\gamma_2;\varphi_1,\varphi_2)$ of normalized analytic functions defined on the open unit disk $\mathfrak{E}$, utilizing a generalized multiplier transformation $\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}$ that is based on the Hadamard product. These subclasses are linked to Ma–Minda-type subordination, which involves functions that map onto the cardioid domain. We investigate several inclusion-type results concerning these subclasses under $\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}$ and examine inclusion results related to the generalized Libera integral operator $\mathcal{F}_{\mathcal{C}}$. Additionally, we emphasize several well-known subclasses of analytic functions that arise as special cases when specific parameter values are assigned.

Keywords: analytic functions; cardioid domain; Choi-Saigo-Srivastava fractional operator; multiplier transformation; integral operator.

1 Introduction

Let $\mathfrak{A}$ represent the class of functions that have the following form;

$$f(\psi) = \psi + \sum_{n=2}^{\infty} a_n \psi^n, \tag{1}$$

analytic in $\mathfrak{E} = \{\psi \in \mathbb{C} : |\psi| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$. If f and g are analytic in $\mathfrak{E}$, we say that f is subordinate to g, and is written as $f \prec g$ if, there exists a function $\mathfrak{u}$ such that $f(\psi) = g(\mathfrak{u}(\psi))$, $\psi \in \mathfrak{E}$, where $\mathfrak{u}(0) = 0$, $|\mathfrak{u}(\psi)| < 1$ for $\psi \in \mathfrak{E}$. Let, for $0 \le \eta < 1$, $S^*(\eta)$, $C(\eta)$, and $\mathcal{K}$ be the subclasses of $\mathfrak{A}$ comprising all analytic functions that are respectively starlike of order η, convex of order η, and close-to-convex within the unit disk $\mathfrak{E}$. For more details, see [21]. For any real number $\mathfrak{s}$, Cho and Srivastava [7] and Cho and Kim [6] introduced the multiplier transformation $\mathfrak{I}_\lambda^{\mathfrak{s}}$ of functions $f \in \mathfrak{A}$ by,

$$\mathfrak{I}_\lambda^{\mathfrak{s}} f(\psi) = \psi + \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^{\mathfrak{s}} a_n \psi^n, (\mathfrak{s} \in \mathbb{R};\ \lambda > -1).$$

The operator is readily seen to satisfy the composition property:

$$\mathfrak{I}_\lambda^{\mathfrak{s}} \left(\mathfrak{I}_\lambda^{\mathfrak{t}} f(\psi)\right) = \mathfrak{I}_\lambda^{\mathfrak{s}+\mathfrak{t}} f(\psi),$$

whenever $\mathfrak{s}$ and $\mathfrak{t}$ are real numbers. The following special cases of this operator have been studied extensively.

- For integer values of $\mathfrak{s}$ and $\lambda = 1$, $\mathfrak{I}_\lambda^{\mathfrak{s}}$ was examined by Uralegaddi and Somanatha [22].
- For $\mathfrak{s} = -1$, it becomes the integral operator investigated by Owa and Srivastava [16].
- For negative real $\mathfrak{s}$ and $\lambda = 1$, it represents a multiplier transformation introduced by Jung et al. [12] (see also, [3]).
- For $\lambda = 0$, the operator $\mathfrak{I}_\lambda^{\mathfrak{s}}$ is the differential operator introduced by Salagean [19].

It is worth noting that the operator $\mathfrak{I}_\lambda^{\mathfrak{s}}$ is closely linked to the multiplier transformation introduced by Flett [10]. In a further generalization, Cho and Kim [5] defined the operator $\mathfrak{I}_{\lambda,\mu}^{\mathfrak{s}}:\mathfrak{A} \to \mathfrak{A}$ by

$$\mathfrak{I}_{\lambda,\mu}^{\mathfrak{s}} f(\psi) = f_{\lambda,\mu}^{\mathfrak{s}}(\psi) * f(\psi), \qquad (\mathfrak{s} \in \mathbb{R}; \lambda > -1; \mu > 0; \psi \in \mathfrak{E}),$$

where $*$ denotes the Hadamard product (or convolution) and $f_{\lambda,\mu}^{\mathfrak{s}}(\psi)$ is the function satisfying

$$f_\lambda^{\mathfrak{s}}(\psi) * f_{\lambda,\mu}^{\mathfrak{s}}(\psi) = \frac{\psi}{(1-\psi)^\mu}.$$

From this definition, it follows that

$$\mathfrak{I}_{0,2}^0(\psi) = \psi f'(\psi) \text{ and } \mathfrak{I}_{0,2}^1(\psi) = f(\psi),$$

for $f \in \mathfrak{A}$, with $\lambda > -1$ and $\mu > 0$, the following differential identities hold:

$$\psi\left(\mathfrak{I}_{\lambda,\mu}^{\mathfrak{s}+1}(\psi)\right)' = (\lambda+1)\,\mathfrak{I}_{\lambda,\mu}^{\mathfrak{s}}(\psi) - \lambda \mathfrak{I}_{\lambda,\mu}^{\mathfrak{s}+1}(\psi), \tag{2}$$

$$\psi\left(\mathfrak{I}_{\lambda,\mu}^{\mathfrak{s}}(\psi)\right)' = \mu \mathfrak{I}_{\lambda,\mu+1}^{\mathfrak{s}}(\psi) - (\mu-1)\,\mathfrak{I}_{\lambda,\mu}^{\mathfrak{s}}(\psi). \tag{3}$$

In recent years, subclasses of univalent and bi-univalent functions defined by various operators have received considerable attention. Specifically, coefficient estimates and associated geometric properties of such functions have been obtained by employing fractional derivatives and linear operators (see, for instance,[18, 14]). On the other hand,[8] examined multivalent Bessel functions associated with a new integral operator and established several interesting results.

Define $\mathfrak{F}$ as the set of analytic and univalent functions φ in $\mathfrak{E}$. Suppose that φ satisfies the normalization $\varphi(0) = 1$ and $\varphi^{'}(0) > 0$, and maps $\mathfrak{E}$ onto a domain that is convex, starlike with respect to the point 1, symmetric with respect to the real axis, and such that,

$$\Re(\varphi(\psi)) > 0, \text{ for all } \psi \in \mathfrak{E}.$$

For such a function φ, Ma and Minda [13] introduced the class $S^*(\varphi)$ defined by

$$S^*(\varphi) = \left\{ f \in \mathfrak{A} : \frac{\psi f^{'}(\psi)}{f(\psi)} \right\} \prec \varphi(\psi).$$

This family is referred to as the Ma–Minda type starlike functions. Additionally, the corresponding class of convex functions is represented by $C(\varphi)$, which is defined using the Alexander relation [2] as follows:

$$f \in C(\varphi) \Leftrightarrow \psi f^{'}(\psi) \in S^*(\varphi).$$

Ravichandran et al. [17] further generalized the Ma–Minda classes by introducing the classes:

$$S^*(\gamma_1; \varphi) = \left\{ f \in \mathfrak{A} : 1 + \frac{1}{\gamma_1} \left(\frac{\psi f^{'}(\psi)}{f(\psi)} - 1 \right) \prec \varphi(\psi) \right\},$$

and

$$C(\gamma_1; \varphi) = \left\{ f \in \mathfrak{A} : 1 + \frac{1}{\gamma_1} \left(\frac{\psi f^{''}(\psi)}{f^{'}(\psi)} \right) \prec \varphi(\psi) \right\},$$

where $\gamma_1 \in \mathbb{C} \backslash \{0\}$. Several specific subfamilies of $S^*(\varphi)$ have been studied. Below are some well-known specific subfamilies of Ma–Minda type starlike functions. If we select $\varphi(\psi) = \dfrac{1 + A\psi}{1 + B\psi}$, where $-1 \leq B < A \leq 1$, we obtain the classes of Janowski starlike functions and Janowski convex functions, denoted as $S^*[A, B]$ and $C[A, B]$, respectively. These were introduced by Janowski [11]. For $\varphi_c(\psi) = 1 + \frac{4}{3}\psi + \frac{2}{3}\psi^2$, Sharma et al. [20] introduced the Ma-Minda family S^*_{car} of analytic functions associated with the cardioid curve defined by the equation

$$\left(9u^2 + 9v^2 - 18v + 5\right)^2 - 16\left(9u^2 + 9v^2 - 6v + 1\right) = 0.$$

The related work can be seen in [1, 4]. Inspired by the developments mentioned above and recent research into subordination principles and operator theory, we introduce new subclasses of analytic functions that are defined in terms of a domain bounded by a cardioid curve.

Definition 1.1. *Let f be a function of the form (1). Then f is said to belong to the subclass $\mathfrak{S}^*_{car}(\gamma_1; \varphi_c)$ of starlike functions of complex order γ_1 if it satisfies the subordination condition:*

$$1 + \frac{1}{\gamma_1} \left(\frac{\psi f^{'}(\psi)}{f(\psi)} - 1 \right) \prec \varphi_c(\psi), \qquad (\psi \in \mathfrak{E}),$$

where $\gamma_1 \in \mathbb{C}\backslash\{0\}$, $\varphi_c(\psi) \in \mathfrak{F}$ *and is given by*

$$\varphi_c(\psi) = 1 + \frac{4}{3}\psi + \frac{2}{3}\psi^2. \tag{4}$$

Alternatively, $f \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c)$ *if the function* $\left(1 + \frac{1}{\gamma_1}\left(\frac{\psi f'(\psi)}{f(\psi)} - 1\right)\right)$, *maps into the cardioid-shaped region defined by* (4).

Definition 1.2. *A function* f *of the form* (1) *is said to belong to the subclass* $\mathfrak{C}_{car}(\gamma_1; \varphi_c)$ *of convex functions of order* γ_1 *associated with the cardioid domain if it satisfies the following condition:*

$$1 + \frac{1}{\gamma_1}\left(\frac{\psi f''(\psi)}{f'(\psi)} - 1\right) \prec \varphi_c(\psi), \qquad (\psi \in \mathfrak{E}),$$

where $\gamma_1 \in \mathbb{C}\backslash\{0\}$ *and* $\varphi_c(\psi)$ *is given by* (4).

Definition 1.3. *A function* f *of the form* (1) *is said to be in the subclass* $\mathfrak{K}_{car}(\gamma_1, \gamma_2; \varphi_1, \varphi_2)$ *of close-to-convex functions associated with cardioid domain if there exists a function* $\mathfrak{g}(\psi) \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c)$, *such that;*

$$1 + \frac{1}{\gamma_2}\left(\frac{\psi f'(\psi)}{\mathfrak{g}(\psi)} - 1\right) \prec \varphi_2(\psi), \qquad (\psi \in \mathfrak{E}),$$

where $\gamma_1, \gamma_1 \in \mathbb{C}\backslash\{0\}$, $\varphi_1, \varphi_2 \in \mathfrak{F}$ *with* $\varphi_2 = \varphi_c$ *given by* (4).

It is worth mentioning that, for particular selections of the functions φ_1, φ_2 and parameters γ_1, γ_2, the newly introduced subclasses simplify to several classical function classes. For example, we have:

Remark 1.1. *Taking* $\gamma_1 = 1 - \eta$, $(0 \leq \eta < 1)$ *and* $\varphi_c = \dfrac{1+\psi}{1-\psi}$, *in Definitions 1.1 and 1.2, we obtain,*

$$\mathfrak{S}^*_{car}(\gamma_1; \varphi_1) \equiv S^*(\eta) \qquad \textit{and} \qquad \mathfrak{C}_{car}(\gamma_1; \varphi_c) \equiv C(\eta),$$

respectively. For $\varphi_1, \varphi_2 \in \mathfrak{F}$ *and* $\gamma_1 = \gamma_2 = 0$, *in Definition 1.3, we get* $\mathfrak{K}_{car}(\gamma_1, \gamma_2; \varphi_1, \varphi_1) \equiv \mathcal{K}$.

Remark 1.2. *For* $\gamma_1 = 1$, *Definitions 1.1 and 1.2 reduce to*

$$\mathfrak{S}^*_{car}(\gamma_1; \varphi_1) \equiv S^*_{car} \qquad \textit{and} \qquad \mathfrak{C}_{car}(\gamma_1; \varphi_c) \equiv C_{car},$$

which were studied by Sharma et al. [20].

Remark 1.3. *For* $\varphi_1, \varphi_2 \in \mathfrak{F}$ $\gamma_1 = 1 - \eta$ *and* $\gamma_2 = 1 - \delta$ $(0 \leq \eta, \delta < 1)$, *in Definitions 1.1, 1.2 and 1.3, we obtain:*

$$\mathfrak{S}^*_{car}(\gamma_1; \varphi_c) \equiv S^*(\eta; \varphi_1), \qquad \mathfrak{C}_{car}(\gamma_1; \varphi_c) \equiv K(\eta; \varphi_1),$$

and

$$\mathfrak{K}_{car}(\gamma_1, \gamma_2; \varphi_1, \varphi_2) \equiv \mathcal{C}(\eta, \delta; \varphi_1, \varphi_2).$$

These subclasses were studied by Cho and Kim [5]**.**

Utilizing the operator $\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}$, we define the following novel subclasses of analytic functions. These subclasses define generalized Ma–Minda type functions of complex order via subordination involving the operator $\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}$.

Definition 1.4. *Let* $\varphi_1 \in \mathfrak{F}$, $\mathfrak{s} \in \mathbb{R}$ *and* $\lambda > -1$, $\mu > 0$, *then for a non-zero complex number* γ_1, *the subclass* $\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1)$ *is defined as*

$$\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\phi_1) = \left\{ f \in \mathfrak{A} : \mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi) \in \mathfrak{S}^{*}_{car}(\gamma_1;\varphi_c) \right\}.$$

Definition 1.5. *Let* $\varphi_1 \in \mathfrak{F}$, $\mathfrak{s} \in \mathbb{R}$, $\gamma_1 \in \mathbb{C}\backslash\{0\}$ *and* $\lambda > -1$, $\mu > 0$, *then the subclass* $\mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1)$ *is defined as*

$$\mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1) = \left\{ f \in \mathfrak{A} : \mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi) \in \mathfrak{C}_{car}(\gamma_1;\varphi_c) \right\}.$$

Definition 1.6. *Let* γ_1 *be a non-zero complex number, then the subclass* $\mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1,\gamma_2;\varphi_1,\varphi_2)$ *is defined as*

$$\mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1,\gamma_2;\varphi_1,\varphi_2) = \left\{ f \in \mathfrak{A} : \mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi) \in \mathfrak{K}_{car}(\gamma_1,\gamma_2;\varphi_1,\varphi_c) \right\},$$

where $\varphi_1,\varphi_2 \in \mathfrak{F}$, $\gamma_1,\gamma_1 \in \mathbb{C}\backslash\{0\}$, $\mathfrak{s} \in \mathbb{R}$ *and* $\lambda > -1$, $\mu > 0$.

We also observe that

$$f(\psi) \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1,\phi_1) \Leftrightarrow \psi f'(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1). \tag{5}$$

Remark 1.4. *For* $\varphi_1,\varphi_2,\varphi_c \in \mathfrak{F}$ *and* $\gamma_1 = 1-\eta$, $\gamma_2 = 1-\delta\ (0 \le \eta,\delta < 1)$, *in Definitions 1.4, 1.5 and 1.6, we obtain the following subclasses of analytic functions investigated by Cho and Kim [5].*
As a particular case, we set

$$\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}\left(\gamma_1;\left(\frac{1+M\psi}{1+N\psi}\right)^{\alpha}\right) = \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;M,N,\alpha), \qquad (0<\alpha\le 1, -1\le N<M\le 1),$$

and

$$\mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}\left(\gamma_1;\left(\frac{1+M\psi}{1+N\psi}\right)^{\alpha}\right) = \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;M,N,\alpha), \qquad (0<\alpha\le 1, -1\le N<M\le 1).$$

Remark 1.5. *For* $\gamma_1 = 1-\eta\ (0 \le \eta < 1)$, *in the above particular case, we obtain the following particular case examine by Cho and Kim [5]:*

$$\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}\left(\gamma_1;\left(\frac{1+M\psi}{1+N\psi}\right)^{\alpha}\right) \equiv S^{\mathfrak{s}}_{\lambda,\mu}\left(\eta;\left(\frac{1+M\psi}{1+N\psi}\right)^{\alpha}\right), \qquad (0<\alpha\le 1, -1\le N<M\le 1),$$

and

$$\mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}\left(\gamma_1;\left(\frac{1+M\psi}{1+N\psi}\right)^{\alpha}\right) \equiv K^{\mathfrak{s}}_{\lambda,\mu}\left(\eta;\left(\frac{1+M\psi}{1+N\psi}\right)^{\alpha}\right), \qquad (0<\alpha\le 1, -1\le N<M\le 1).$$

2 Preliminaries

For our investigation, the following results are needed.

Lemma 2.1. [9]. *Assume that* φ_1 *is a convex univalent function in* $\mathfrak{E}$ *satisfying* $\varphi_1(0) = 1$. *Suppose that there exist complex constants* $\mathsf{k},\mathsf{v} \in \mathbb{C}$ *such that*

$$\Re\{\mathsf{k}\varphi_1(\psi) + \mathsf{v}\} > 0.$$

Furthermore, let p *be an analytic function in* $\mathfrak{E}$ *with* $\mathsf{p}(0) = 1$, *then*

$$\mathsf{p}(\psi) + \frac{\psi \mathsf{p}'(\psi)}{\mathsf{k}\mathsf{p}(\psi)} \prec \varphi_1(\psi),$$

implies that

$$\mathsf{p}(\psi) \prec \varphi_1(\psi)\,(\psi \in \mathfrak{E}).$$

Lemma 2.2. [15]. *Assume that φ_1 is a convex univalent function in $\mathfrak{E}$, and let $\mathfrak{w}$ be analytic in $\mathfrak{E}$ such that*

$$\Re\{\mathfrak{w}(\psi)\} \geq 0.$$

Furthermore, let p be an analytic function in $\mathfrak{E}$ with $\mathsf{p}(0) = 1$, then

$$\mathsf{p}(\psi) + \mathfrak{w}(\psi)\,\psi\mathsf{p}'(\psi) \prec \varphi_1(\psi),$$

implies that

$$\mathsf{p}(\psi) \prec \varphi_1(\psi)\,(\psi \in \mathfrak{E}).$$

3 Main Results

In this section, we examine various inclusion properties of the subclasses $\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1)$, $\mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1)$ and $\mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1,\gamma_2;\varphi_1,\varphi_2)$ associated with the operator $\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}$, and investigate inclusion results related to the generalized Libera integral operator $\mathcal{F}_{\mathcal{C}}$.Additionally, we highlight several well-known subclasses of analytic functions obtained as special cases by assigning specific parameter values.

3.1 The operator $\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}$ and its inclusion-related properties

Theorem 3.1. *Let $\mathfrak{s} \in \mathbb{R}$, with λ taken from the non-negative reals and and $\mu \geq 1$. Then the following holds:*

$$\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1;\varphi_1) \subset \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1) \subset \mathfrak{S}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1;\varphi_1), \qquad (\gamma_1 \in \mathbb{C}\setminus\{0\};\varphi_1 \in \mathfrak{F}).$$

Proof. Let us first show that

$$\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1;\varphi_1) \subset \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1).$$

Let $f \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1;\varphi_1)$ and set

$$\mathsf{p}(\psi) = 1 + \frac{1}{\gamma_1}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)} - 1\right). \tag{6}$$

Here, $\mathsf{p}(\psi)$ is analytic in $\mathfrak{E}$ and $\mathsf{p}(0) = 1$, it follows from (3) and (1.6) that

$$\mu\frac{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1} f(\psi)}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)} = \gamma_1\mathsf{p}(\psi) + \mu - \gamma_1. \tag{7}$$

Differentiating logarithmically on both sides of (7) and multiply by ψ, we obtain the following:

$$1 + \frac{1}{\gamma_1}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1} f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1} f(\psi)} - 1\right) = \mathsf{p}(\psi) + \frac{\psi f'(\psi)}{\gamma_1\mathsf{p}(\psi) + \mu - \gamma_1}. \tag{8}$$

By using Lemma 2.1 to (8), we conclude that $\mathsf{p} \prec \varphi_1$; that is, $f \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1)$. Proceeding to the second part, let us assume $f \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1)$ and define

$$\mathsf{q}(\psi) = 1 + \frac{1}{\gamma_1}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}+1}_{\lambda,\mu} f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}+1}_{\lambda,\mu} f(\psi)} - 1\right),$$

where the function $\mathsf{q}(\psi)$ is analytic in the open unit disk $\mathfrak{E}$ with $\mathsf{q}(0) = 1$. Subsequently, using arguments analogous to those discussed above with (2), we infer that $\mathsf{q}(\psi) \prec \varphi_1$ in $\mathfrak{E}$, which implies that $f \in \mathfrak{S}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1; \varphi_1)$. We have thus completed the proof of Theorem 3.1. □

Remark 3.1. *For $\gamma_1 = 1 - \eta$ $(0 \le \eta < 1)$ in Theorem 3.1, we obtain the following known result (Theorem 2.1, p. 326) for the subclass $S^{\mathfrak{s}}_{\lambda,\mu}(\eta; \varphi_1)$, studied by [5].*

Theorem 3.2. [5]. *Let $\mathfrak{s} \in \mathbb{R}$, $\mu \ge 1$ and $\lambda \ge 0$. Then*

$$S^{\mathfrak{s}}_{\lambda,\mu+1}(\eta; \varphi_1) \subset S^{\mathfrak{s}}_{\lambda,\mu}(\eta; \varphi_1) \subset S^{\mathfrak{s}+1}_{\lambda,\mu}(\eta; \varphi_1), \qquad (0 \le \eta < 1; \varphi_1 \in \mathfrak{F}).$$

Theorem 3.3. *Let $\mathfrak{s} \in \mathbb{R}$, $\mu \ge 1$ and $\lambda \ge 0$. Then*

$$\mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1; \varphi_1) \subset \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) \subset \mathfrak{C}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1; \varphi_1), \qquad (\gamma_1 \in \mathbb{C}\setminus\{0\}; \varphi_1 \in \mathfrak{F}).$$

Proof. Using (5) along with Theorem 3.1, we note that

$$\begin{aligned}
f(\psi) \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1; \varphi_1) &\Leftrightarrow \mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1} f(\psi) \in \mathfrak{C}_{car}(\gamma_1; \varphi_c),\\
\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1} f(\psi) \in \mathfrak{C}_{car}(\gamma_1; \varphi_c) &\Leftrightarrow \psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1} f(\psi)\right)' \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c),\\
\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1} f(\psi)\right)' \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c) &\Leftrightarrow \mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1}\left(\psi f'(\psi)\right) \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c),\\
\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1}\left(\psi f'(\psi)\right) \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c) &\Leftrightarrow \psi f'(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1; \varphi_1),\\
\psi f'(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1; \varphi_1) &\Leftrightarrow \psi f'(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1),\\
\psi f'(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) &\Leftrightarrow \mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\left(\psi f'(\psi)\right) \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c),\\
\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\left(\psi f'(\psi)\right) \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c) &\Leftrightarrow \psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)\right)' \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c),\\
\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)\right)' \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c) &\Leftrightarrow \mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi) \in \mathfrak{C}_{car}(\gamma_1; \varphi_c),\\
\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi) \in \mathfrak{C}_{car}(\gamma_1; \varphi_c) &\Leftrightarrow f(\psi) \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1).
\end{aligned}$$

Finally, by assembling these equivalences, we obtain the following:

$$f(\psi) \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1; \varphi_1) \Leftrightarrow f(\psi) \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1),$$

implies that

$$\mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1; \varphi_1) \subset \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1).$$

Now, for the second part, again using (5), we have

$$\begin{aligned}
f(\psi) \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) &\Leftrightarrow \psi f'(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1),\\
&\Rightarrow \psi f'(\psi) \in \mathfrak{S}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1; \varphi_1),\\
\psi f'(\psi) \in \mathfrak{S}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1; \varphi_1) &\Leftrightarrow \psi(\mathfrak{I}^{\mathfrak{s}+1}_{\lambda,\mu} f(\psi))' \in \mathfrak{S}^*_{car}(\gamma 1; \varphi_c),\\
\psi(\mathfrak{I}^{\mathfrak{s}+1}_{\lambda,\mu} f(\psi))' \in \mathfrak{S}^*_{car}(\gamma_1; \varphi_c) &\Leftrightarrow \mathfrak{I}^{\mathfrak{s}+1}_{\lambda,\mu} f(\psi) \in \mathfrak{C}_{car}(\gamma_1; \varphi_c),\\
\mathfrak{I}^{\mathfrak{s}+1}_{\lambda,\mu} f(\psi) \in \mathfrak{C}_{car}(\gamma_1; \varphi_c) &\Leftrightarrow f(\psi) \in \mathfrak{C}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1; \varphi_1),
\end{aligned}$$

Thus, it follows that

$$f(\psi) \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) \Leftrightarrow f(\psi) \in \mathfrak{C}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma; \varphi_1).$$

Therefore, the inclusion $\mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) \subset \mathfrak{C}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1; \varphi_1)$ holds, and the proof of Theorem 3.3 is complete. □

If we choose

$$\varphi_1(\psi) = \left(\frac{1+M\psi}{1+N\psi}\right)^{\alpha}, \qquad (0 < \alpha \le 1; -1 \le N < M \le 1; \psi \in \mathfrak{E}),$$

then an application of Theorems 3.1, and 3.3 leads to the following corollary.

Corollary 3.1. *Let $\mathfrak{s} \in \mathbb{R}, \mu \ge 1$ and $\lambda \ge 0$. Then*

$$\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1; M, N; \alpha) \subset \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; M, N; \alpha) \subset \mathfrak{S}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1; M, N; \alpha),$$

and

$$\mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1; M, N; \alpha) \subset \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; M, N; \alpha) \subset \mathfrak{C}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1; M, N; \alpha),$$

where $\alpha \in (0,1], \gamma_1 \in \mathbb{C}\setminus\{0\}$, and the parameters N and M satisfy $-1 \le N < M \le 1$.

Remark 3.2. *For $\varphi_1, \varphi_c \in \mathfrak{F}$ and $\gamma_1 = 1-\eta$ $(0 \le \eta < 1)$. Utilizing Theorem 3.3, this yields a previously established result for the subclass $K^{\mathfrak{s}}_{\lambda,\mu}(\eta; \varphi_1)$, investigated by Cho and Kim [5].*

Theorem 3.4. [5]. *Let $\mathfrak{s} \in \mathbb{R}, \mu \ge 1$ and $\lambda \ge 0$. Then*

$$K^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1; \varphi_1) \subset K^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) \subset K^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1; \varphi_1)\,(0 \le \eta < 1; \varphi_1 \in \mathfrak{F}).$$

Remark 3.3. *For $\gamma_1 = 1-\eta$ with $(0 \le \eta < 1)$, Corollary 3.1, yields the following known result for the subclasses $S^{\mathfrak{s}}_{\lambda,\mu}(\eta; \varphi_1)$ and $K^{\mathfrak{s}}_{\lambda,\mu}(\eta;\varphi_1)$,which were previously studied by Cho and Kim [5].*

Corollary 3.2. [5]. *Let $\mathfrak{s} \in \mathbb{R}, \mu \ge 1$ and $\lambda \ge 0$. Then*

$$S^{\mathfrak{s}}_{\lambda,\mu+1}(\eta; M, N; \alpha) \subset S^{\mathfrak{s}}_{\lambda,\mu}(\eta; M, N; \alpha) \subset S^{\mathfrak{s}+1}_{\lambda,\mu}(\eta; M, N; \alpha)$$

and

$$K^{\mathfrak{s}}_{\lambda,\mu+1}(\eta; M, N; \alpha) \subset K^{\mathfrak{s}}_{\lambda,\mu}(\eta; M, N; \alpha) \subset K^{\mathfrak{s}+1}_{\lambda,\mu}(\eta; M, N; \alpha),$$

where $0 < \alpha \le 1$, $0 \le \eta < 1$ and $-1 \le N < M \le 1$. Finally, applying Lemma 2.2, we derive the following inclusion result for the subclass $\mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1, \gamma_2; \varphi_1, \varphi_2)$.

Theorem 3.5. *Let $\mathfrak{s} \in \mathbb{R}, \mu \ge 1$ and $\lambda \ge 0$. Then*

$$\mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1, \gamma_1; \varphi_1, \varphi_2) \subset \mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1, \gamma_1; \varphi_1, \varphi_2) \subset \mathfrak{K}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1, \gamma_1; \varphi_1, \varphi_2),$$

where $\gamma_1, \gamma_1 \in \mathbb{C}\setminus\{0\}$ and $\varphi_1, \varphi_2 \in \mathfrak{F}$.

Proof. Initially, let us prove that

$$\mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1,\gamma_1;\varphi_1,\varphi_2) \subset \mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1,\gamma_1;\varphi_1,\varphi_2).$$

Let $f \in \mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1,\gamma_1;\varphi_1,\varphi_2)$. Then, based on the definition of the given subclass, there exists a function $\mathfrak{g}(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1;\varphi_1)$ satisfying

$$1+\frac{1}{\gamma_2}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1}f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1}\mathfrak{g}(\psi)}-1\right) \prec \varphi_2.$$

Next, let

$$\mathsf{p}(\psi)=1+\frac{1}{\gamma_2}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)}-1\right),$$

where p is analytic in $\mathfrak{E}$ and satisfies $\mathsf{p}(0)=1$. By applying (3), we obtain

$$(\gamma_2\mathsf{p}(\psi)-\gamma_2+1)\,\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)+(\mu-1)\,\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}f(\psi)=\mu\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1}f(\psi). \tag{9}$$

Upon differentiating (9), and multiplying the result by ψ, we obtain

$$\gamma_2\psi\mathsf{p}'(\psi)\,\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)+(\gamma_2\mathsf{p}(\psi)-\gamma_2+1)\,\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)\right)'=\mu\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1}f(\psi)\right)'-(\mu-1)\,\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}f(\psi)\right)'. \tag{10}$$

Since $\mathfrak{g}(\psi)$ is a member of the subclass $\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu+1}(\gamma_1;\varphi_1)$, it follows directly from Theorem 3.1 that $\mathfrak{g}(\psi)$ also belongs to $\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1)$. Let,

$$\mathsf{h}(\psi)=1+\frac{1}{\gamma_1}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)}-1\right).$$

Proceeding with (3) again, we get

$$\mu\frac{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1}\mathfrak{g}(\psi)}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)}=\gamma_1\mathsf{h}(\psi)-\gamma_1+\mu. \tag{11}$$

By making use of (10) and (11), we derive

$$1+\frac{1}{\gamma_2}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1}f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu+1}\mathfrak{g}(\psi)}-1\right)=\mathsf{p}(\psi)+\frac{\psi\mathsf{p}'(\psi)}{\gamma_1\mathsf{h}(\psi)-\gamma_1+\mu}.$$

Given $\mu \geq 0$ and $\mathsf{h}(\psi) \prec \varphi_1$ in $\mathfrak{E}$, we conclude

$$\Re\{\gamma_1\mathsf{h}(\psi)-\gamma_1+\mu\}>0 \quad (\psi\in\mathfrak{E}).$$

Hence, by applying Lemma 2.2, it follows that $\mathsf{p} \prec \varphi_2$, and therefore $f \in \mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1,\gamma_1;\varphi_1,\varphi_2)$. By repeating the previous steps with reference to (3), we get inclusion results for the second part,

$$\mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1,\gamma_2;\varphi_1,\varphi_2) \subset \mathfrak{K}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1,\gamma_1;\varphi_1,\varphi_2).$$

Thus, the proof of Theorem 3.5 is complete. □

Remark 3.4. *For $\varphi_1, \varphi_2 \in \mathfrak{F}$ and $\gamma_1 = 1-\eta$, $\gamma_2 = 1-\delta$ $(0 \leq \eta, \delta < 1)$ in Theorem 3.5, the following inclusion results hold in connection with $\mathcal{C}^{\mathfrak{s}}_{\lambda,\mu}(\eta,\delta;\varphi_1,\varphi_2)$ examined by Cho and Kim [5].*

Theorem 3.6. *Let the real parameter $\mathfrak{s}$ and the non-negative constants λ and μ satisfy $\mathfrak{s} \in \mathbb{R}$, $\lambda \geq 0$, and $\mu \geq 1$. Then,*

$$\mathcal{C}^{\mathfrak{s}}_{\lambda,\mu+1}(\eta,\delta;\varphi_1,\varphi_2) \subset \mathcal{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\phi_1) \subset \mathcal{C}^{\mathfrak{s}+1}_{\lambda,\mu}(\gamma_1;\phi_1), \qquad (0 \leq \eta,\delta < 1; \varphi_1,\varphi_2 \in \mathfrak{F}).$$

3.2 The integral operator $\mathcal{F}_{\mathcal{C}}$ and its inclusion-related properties

This section focuses on the generalized Libera integral operator $\mathcal{F}_{\mathcal{C}}$ [16], defined by

$$\mathcal{F}_{\mathcal{C}}(f)(\psi) = \frac{\mathsf{c}+1}{\psi^{\mathsf{c}}} \int_0^{\psi} \mathsf{t}^{\mathsf{c}-1} f(\mathsf{t})\, d\mathsf{t}, \text{ where } f \in \mathfrak{A} \text{ and } \mathsf{c} > -1. \tag{12}$$

The proof begins with the following result.

Theorem 3.7. *Let $\mathfrak{s} \in \mathbb{R}$, $\gamma_1 \in \mathbb{C} \setminus \{0\}$ and $\mathsf{c}, \lambda, \mu \in [0,\infty)$. If $f \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1)$, then,*

$$\mathcal{F}_{\mathcal{C}}(f) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1), \qquad (\varphi_1 \in \mathfrak{F}).$$

Proof. Let f belong to the subclass $\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1)$, and let us set

$$\mathsf{p}(\psi) = 1 + \frac{1}{\gamma_1}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_{\mathcal{C}}(f)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_{\mathcal{C}}(f)} - 1\right). \tag{13}$$

Here, p is analytic in $\mathfrak{E}$ and satisfying $\mathsf{p}(0) = 1$. From equation (12), we obtain

$$\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_{\mathcal{C}}(f)\psi\right)' = (\mathsf{c}+1)\,\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi) - \mathsf{c}\,\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_{\mathcal{C}}(f). \tag{14}$$

Next, using (13) and (14), we get

$$(\mathsf{c}+1)\frac{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_{\mathcal{C}}(f)(\psi)} = \gamma_1 \mathsf{p}(\psi) - \gamma_1 + 1 + \mathsf{c}. \tag{15}$$

Applying the principle of logarithmic differentiation to both sides of (15) and multiplying by ψ, we get

$$\mathsf{p}(\psi) + \frac{\psi \mathsf{p}'(\psi)}{\gamma_1 \mathsf{p}(\psi) - \gamma_1 + 1 + \mathsf{c}} = 1 + \frac{1}{\gamma_1}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)} - 1\right).$$

Thus, by applying Lemma 2.1, it follows that $\mathsf{p} \prec \varphi_1$ in $\mathfrak{E}$, which implies that $\mathcal{F}_{\mathcal{C}}(f) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1)$. □

Remark 3.5. *For $\gamma_1 = 1-\eta$ $(0 \leq \eta < 1)$, in Theorem 3.7, we obtain the following known result studied by Cho and Kim [5].*

Theorem 3.8. *Let $\mathfrak{s} \in \mathbb{R}$ and $\mathsf{c}, \lambda \geq 0$, $\mu > 0$. If $f \in S^{\mathfrak{s}}_{\lambda,\mu}(\eta;\varphi_1)$, then*

$$\mathcal{F}_{\mathcal{C}}(f) \in S^{\mathfrak{s}}_{\lambda,\mu}(\eta;\varphi_1), \qquad (\varphi_1 \in \mathfrak{F}; 0 \leq \eta < 1).$$

Proceeding further, we derive the following inclusion property involving $\mathcal{F}_{\mathcal{C}}$.

Theorem 3.9. *Let* $\mathfrak{s} \in \mathbb{R}$ *and* $\mathsf{c}, \lambda \geq 0$, $\mu > 0$. *If* $f \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \phi_1)$, *then*

$$\mathcal{F}_{\mathcal{C}}(f) \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \phi_1), \qquad (\gamma_1 \in \mathbb{C} \backslash \{0\}; \varphi_1 \in \mathfrak{F}).$$

Proof. From Theorem 3.1, we have

$$\begin{aligned} f \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) &\Leftrightarrow \psi f'(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1), \\ \psi f'(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) &\Leftrightarrow \mathcal{F}_{\mathcal{C}}\left(\psi f'(\psi)\right) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1), \\ \mathcal{F}_{\mathcal{C}}\left(\psi f'(\psi)\right) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) &\Leftrightarrow \psi\left(\mathcal{F}_{\mathcal{C}}(f)(\psi)\right)' \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1), \\ \psi\left(\mathcal{F}_{\mathcal{C}}(f)(\psi)\right)' \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) &\Leftrightarrow \mathcal{F}_{\mathcal{C}}(f)(\psi) \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1). \end{aligned}$$

Thus, we obtain

$$f \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1) \Leftrightarrow \mathcal{F}_{\mathcal{C}}(f)(\psi) \in \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1).$$

This completes the proof of Theorem 3.9. □

Remark 3.6. *For* $\gamma_1 = 1 - \eta$ $(0 \leq \eta < 1)$, *in Theorem 3.9, the following known result, studied by [5], can be obtained.*

Theorem 3.10. [5]. *Let* $\mathfrak{s} \in \mathbb{R}$ *and* $\mathsf{c}, \lambda \geq 0$, $\mu > 0$. *If* $f \in K^{\mathfrak{s}}_{\lambda,\mu}(\eta; \varphi_1)$, *then,*

$$\mathcal{F}_{\mathcal{C}}(f) \in K^{\mathfrak{s}}_{\lambda,\mu}(\eta; \varphi_1)\,(0 \leq \eta < 1; \varphi_1 \in \mathfrak{F}).$$

Based on Theorems 3.1 and 3.3, we have the following:

Corollary 3.3. *Let* $\mathfrak{s} \in \mathbb{R}$ *and* $\mathsf{c}, \lambda \geq 0$, $\mu > 0$. *If*

$$f \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; A_1, A_2; \alpha)\left(\text{or } \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; A_1, A_2; \alpha)\right), \qquad (\gamma_1 \in \mathbb{C} \backslash \{0\}; -1 \leq A_1 < A_2 \leq 1; 0 < \alpha \leq 1),$$

then

$$\mathcal{F}_{\mathcal{C}}(f) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; A_1, A_2; \alpha)\left(\text{or } \mathfrak{C}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; A_1, A_2; \alpha)\right).$$

Remark 3.7. *By choosing* $\gamma_1 = 1 - \eta$ *with* $0 \leq \eta < 1$ *in Corollary 3.3 leads to a previously established result presented in the work of* [5].

Corollary 3.4. [5]. *Let* $\mathfrak{s} \in \mathbb{R}$ *and* $\mathsf{c}, \lambda \geq 0$, $\mu > 0$. *If*

$$f \in S^{\mathfrak{s}}_{\lambda,\mu}(\eta; A_1, A_2; \alpha)\left(\text{or } K^{\mathfrak{s}}_{\lambda,\mu}(\eta; A_1, A_2; \alpha)\right), \qquad (0 \leq \eta < 1; -1 \leq A_1 < A_2 \leq 1; 0 < \alpha \leq 1),$$

then

$$\mathcal{F}_{\mathcal{C}}(f) \in S^{\mathfrak{s}}_{\lambda,\mu}(\eta; A_1, A_2; \alpha)\left(\text{or } K^{\mathfrak{s}}_{\lambda,\mu}(\eta; A_1, A_2; \alpha)\right).$$

We now proceed to the final result of this section, as follows:

Theorem 3.11. *Let* $\mathfrak{s} \in \mathbb{R}$ *and* $\mathsf{c}, \lambda \geq 0$, $\mu > 0$. *If* $f \in \mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1, \gamma_2; \varphi_1, \varphi_2)$, *then*

$$\mathcal{F}_{\mathcal{C}}(f) \in \mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1, \gamma_2; \varphi_1, \varphi_2), \qquad (\gamma_1, \gamma_1 \in \mathbb{C} \backslash \{0\}; \varphi_1, \varphi_2 \in \mathfrak{F}).$$

Proof. Let $f \in \mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1, \gamma_2; \varphi_1, \varphi_2)$ .Then, by definition, there exists a function $\mathfrak{g}(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1)$ such that

$$1 + \frac{1}{\gamma_2}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)} - 1\right) \prec \varphi_2(\psi), \qquad (\psi \in \mathfrak{E}).$$

Let

$$\mathsf{p}(\psi) = 1 + \frac{1}{\gamma_2}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(f)(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)} - 1\right),$$

here, p denotes an analytic function in $\mathfrak{E}$ with $\mathsf{p}(0) = 1$. Since

$$\mathfrak{g}(\psi) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1),$$

it follows from Theorem 3.7 that: $\mathcal{F}_C(\mathfrak{g}) \in \mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1; \varphi_1)$. Using (14), we get

$$(\mathsf{c}+1)\,\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi) = (\gamma_2\mathsf{p}(\psi) - \gamma_2 + 1)\,\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi) + \mathsf{c}\,\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(f)(\psi).$$

Upon differentiating and then multiplying by ψ, we obtain

$$\begin{aligned}(\mathsf{c}+1)\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)} &= \gamma_2\psi\mathsf{p}'(\psi) + (\gamma_2\mathsf{p}(\psi) - \gamma_2 + 1)\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)} \\ &\quad + \mathsf{c}\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(f)(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)}.\end{aligned} \tag{16}$$

By using the following relations in (16),

$$\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)} = \gamma_1\mathsf{q}(\psi) - \gamma_1 + 1, \; \frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(f)(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)} = \gamma_2\mathsf{p}(\psi) - \gamma_2 + 1,$$

where

$$\mathsf{q}(\psi) = \frac{1}{\gamma_1}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)} + (\gamma_1 - 1)\right),$$

$$\mathsf{p}(\psi) = \frac{1}{\gamma_2}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(f)(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)} + (\gamma_2 - 1)\right).$$

Equation (16) can be written as,

$$(\mathsf{c}+1)\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu} f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_C(\mathfrak{g})(\psi)} = (\gamma_2\mathsf{p}(\psi) - \gamma_2 + 1)(\gamma_1\mathsf{q}(\psi) - \gamma_1 + 1 + \mathsf{c}) + \gamma_2\psi\mathsf{p}'(\psi).$$

Let

$$(\mathsf{c}+1)\,\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)}=\frac{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_{\mathcal{C}}(\mathfrak{g})(\psi)}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)}\left\{(\gamma_2\mathsf{p}(\psi)-\gamma_2+1)(\gamma_1\mathsf{q}(\psi)-\gamma_1+1+\mathsf{c})+\gamma_2\psi\mathsf{p}'(\psi)\right\}. \tag{17}$$

Furthermore, relation (15) for the function $\mathfrak{g}\in\mathfrak{S}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1;\varphi_1)$ with $\mathsf{p}(\psi)=\mathsf{q}(\psi)$, is given by

$$\frac{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathcal{F}_{\mathcal{C}}(\mathfrak{g})(\psi)}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)}=\frac{(\mathsf{c}+1)}{\gamma_1\mathsf{q}(\psi)-\gamma_1+1+\mathsf{c}}.$$

Therefore, (17) reduces to

$$\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)}-(\gamma_2-1)=\gamma_2\mathsf{p}(\psi)+\frac{\gamma_2\psi\mathsf{p}'(\psi)}{\gamma_1\mathsf{q}(\psi)-\gamma_1+1+\mathsf{c}}.$$

Dividing γ_2, on both sides we get

$$1+\frac{1}{\gamma_2}\left(\frac{\psi\left(\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}f(\psi)\right)'}{\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}\mathfrak{g}(\psi)}-1\right)=\mathsf{p}(\psi)+\frac{\psi\mathsf{p}'(\psi)}{\gamma_1\mathsf{q}(\psi)-\gamma_1+1+\mathsf{c}}.$$

Lemma 2.1 leads to the conclusion that $\mathsf{p}\prec\varphi_2$, and consequently $\mathcal{F}_{\mathcal{C}}(f)\in\mathfrak{K}^{\mathfrak{s}}_{\lambda,\mu}(\gamma_1,\gamma_2;\varphi_1,\varphi_2)$. The remaining part of the proof in Theorem 3.11 is analogous to that of Theorem 3.5, the result follows from similar steps and is omitted. □

Remark 3.8. *For $\gamma_1=1-\eta$, $\gamma_2=1-\delta$ $(0\le\eta,\delta<1)$ in Theorem 3.11, we get the following known inclusion results, for the subclass $\mathcal{C}^{\mathfrak{s}}_{\lambda,\mu}(\eta,\delta;\varphi_1,\varphi_2)$ investigated by Cho and Kim [5].*

Theorem 3.12. *Let $\mathfrak{s}\in\mathbb{R}$ and $\mathsf{c},\lambda\ge 0$, $\mu>0$. If $f\in\mathcal{C}^{\mathfrak{s}}_{\lambda,\mu}(\eta,\delta;\varphi_1,\varphi_2)$, then*

$$\mathcal{F}_{\mathcal{C}}(f)\in\mathcal{C}^{\mathfrak{s}}_{\lambda,\mu}(\eta,\delta;\varphi_1,\varphi_2)\qquad(0\le\eta,\delta<1;\varphi_1,\varphi_2\in\mathfrak{F}).$$

4 Conclusions

In this paper, new subclasses of normalized analytic via the generalized multiplier transformation $\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}$, based on the Hadamard product are introuduced. These subclasses are defined through Ma–Mninda-type subordination technique involving functions mapping onto the cardioid domain. Various inclusion relationships among these subclasses under the action of the operator $\mathfrak{I}^{\mathfrak{s}}_{\lambda,\mu}$ are investigated. Furthermore, the inclusion properties of these subclasses in relation to the generalized Libera integral operator $\mathcal{F}_{\mathcal{C}}$ are examined. Our study also demonstrated several well-known subclasses of analytic functions as special cases by specifying appropriate parameter values.

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References

[1] A. Ahmed, A. W. Shaikh & S. A. Shah (2025). Fekete-Szegö problem for certain analytic functions of complex order associated with cardioid domain. *International Journal of Analysis and Applications*, *23*, 109–109. https://doi.org/10.28924/2291-8639-23-2025-109.

[2] J. W. Alexander (1915). Functions which map the interior of the unit circle upon simple regions. *Annals of Mathematics*, *17*(1), 12–22. https://doi.org/10.2307/2007212.

[3] S. D. Bernardi (1969). Convex and starlike univalent functions. *Transactions of the American Mathematical Society*, *135*, 429–446.

[4] A. A. Bhatti, A. W. Shaikh & S. A. Shah (2025). Coefficient bounds for certain classes of analytic functions of complex order γ associated with cardioid domain. *Proceedings of the Pakistan Academy of Sciences: A. Physical and Computational Sciences*, *62*(2), 131–136. https://doi.org/10.53560/PPASA(62-2)860.

[5] N. E. Cho & J. A. Kim (2006). Inclusion properties of certain subclasses of analytic functions defined by a multiplier transformation. *Computers and Mathematics with Applications*, *52*(3–4), 323–330. https://doi.org/10.1016/j.camwa.2006.08.022.

[6] N. E. Cho & T. H. Kim (2003). Multiplier transformations and strongly close-to-convex functions. *Bulletin of the Korean Mathematical Society*, *40*(3), 399–410. https://doi.org/10.4134/bkms.2003.40.3.399.

[7] N. E. Cho & H. M. Srivastava (2003). Argument estimates of certain analytic functions defined by a class of multiplier transformations. *Mathematical and Computer Modelling*, *37*(1–2), 39–49. https://doi.org/10.1016/s0895-7177(03)80004-3.

[8] E. O. Davids, O. A. Fadipe-Joseph & M. O. Oluwayemi (2025). Results on multivalent Bessel functions associated with a new integral operator. *Partial Differential Equations in Applied Mathematics*, *14*, Article ID: 101228. https://doi.org/10.1016/j.padiff.2025.101228.

[9] P. Eenigenburg, P. T. Mocanu, S. S. Miller & M. O. Reade (1983). On a Briot-Bouquet differential subordination. In *General Inequalities 3, International Series of Numerical Mathematics, Vol. 64*, pp. 339–348. Birkhäuser Verlag, Basel, Switzerland.

[10] T. M. Flett (1972). The dual of an inequality of Hardy and Littlewood and some related inequalities. *Journal of Mathematical Analysis and Applications*, *38*(3), 746–765. https://doi.org/10.1016/0022-247x(72)90081-9.

[11] W. Janowski (1970). Extremal problems for a family of functions with positive real part and for some related families. *Annales Polonici Mathematici*, *23*(2), 159–177.

[12] I. B. Jung, Y. C. Kim & H. M. Srivastava (1993). The Hardy space of analytic functions associated with certain one-parameter families of integral operators. *Journal of Mathematical Analysis and Applications*, *176*(1), 138–147. https://doi.org/10.1006/jmaa.1993.1204.

[13] W. Ma & D. Minda (1992). A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis*, pp. 157–169. International Press, Cambridge, USA.

[14] A. N. Metkari, N. D. Sangle & M. Darus (2022). A new subclass of harmonic univalent functions defined by a linear operator. *Malaysian Journal of Mathematical Sciences*, *16*(3), 471–481. https://doi.org/10.47836/mjms.16.3.05.

[15] S. S. Miller & P. T. Mocanu (1981). Differential subordinations and univalent functions. *Michigan Mathematical Journal*, *28*(2), 157–172. https://doi.org/10.1307/mmj/1029002507.

[16] S. Owa & H. M. Srivastava (1986). Some applications of the generalized Libera integral operator. *Proceedings of the Japan Academy, Series A*, *62*(4), 125–128. https://doi.org/10.3792/pjaa.62.125.

[17] V. Ravichandran, Y. Polatoglu, M. Bolcal & A. Sen (2005). Certain subclasses of starlike and convex functions of complex order. *Hacettepe Journal of Mathematics and Statistics*, *34*(1), 9–15.

[18] F. M. Sakar & H. O. Guney (2017). Faber polynomial coefficient estimates for subclasses of m-Fold symmetric Bi-univalent functions defined by fractional derivative. *Malaysian Journal of Mathematical Sciences*, *11*(2), 275–287.

[19] G. S. Salagean (2006). Subclasses of univalent functions. In *Complex Analysis–Fifth Romanian-Finnish Seminar*, pp. 362–372. Springer, Berlin and Heidelberg, Germany. https://doi.org/10.1007/BFb0066543.

[20] K. Sharma, N. K. Jain & V. Ravichandran (2016). Starlike functions associated with a cardioid. *Afrika Matematika*, *27*(5), 923–939. https://doi.org/10.1007/s13370-015-0387-7.

[21] H. M. Srivastava & S. Owa (1992). *Current topics in analytic function theory*. World Scientific Publishing Company, Singapore. https://doi.org/10.1142/1628.

[22] B. A. Uralegaddi & C. Somanatha (1992). Certain classes of univalent functions. In *Current Topics in Analytic Function Theory*, pp. 371–374. World Scientific Publishing Company, Singapore.